A FINANCIAL ENGINEERING VIEW OF DRAWDOWNS IN THE STOCK MARKET PART I

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ABSTRACT. In this report, we collect some topics about how the stochastic (probabilistic) model analysis can provide us with insight into stock investment. We focus on drawdowns from the running maximum. Suppose that we start observing the stock price at time 0, and after a certain time interval t, the highest price between 0 and t has been attained at, say, s. The price at time s is called the running maximum up to time t. It follows that the current price, which is the price at time t, is lower than the running maximum. The stock is said to be in a *drawdown* period (from the running maximum).

In financial engineering, for evaluating the price and risk of financial options, the underlying asset (like stock) is modeled as a stochastic process. While we cannot tell exactly where the process goes at the next moment, we know its distribution: with certain probabilities it goes to certain places. Using this information, we can make some analysis about its path properties. In Part I, we shall discuss financial modeling of drawdown process by taking the Nikkei 225 index as an example. Part II will follow where we look into two more problems of interest. We attempt to avoid technical details in favor of intuitive understanding.

1. INTRODUCTION

Let us denote the stock price or index by $X = \{X_t; t \geq 0\}$, which moves as time goes. One of the quantities in which investors are interested is the running maximum of X, that is, the record high up to certain time t:

$$M_t := \max_{0 \leq s \leq t} X_s.$$

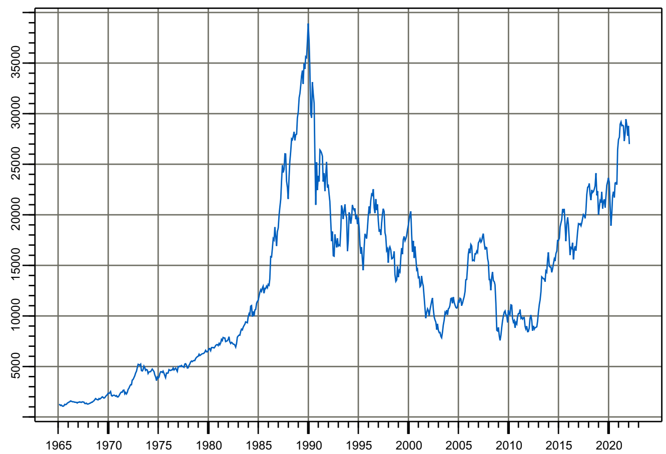
Figure 1 shows the Nikkei 225 index from January 1965 to January 2022. It hit a historical high on December 29, 1989 at 38,915.87. We could, of course, freely set a first date ($t = 0$) as we wish, when we discuss a maximum value. For example, if we set the first date on January 4 in 2000, then the maximum up to January 31, 2022 was attained on September 14, 2021 at 30,670.10.[1]

We define

$$Y_t := M_t - X_t, \tag{1.1}$$

[1] We obtained the data from Refinitiv Eikon.

Fig. 1. The Nikkei 225 monthly data from January 1965 to January 2022.



which is called the *excursion* of X from the running maximum M. It is also called *drawdown* in financial markets. This is the main object in this report. We shall discuss in detail (with some illustrations) in the next section.

The maximum value appears also in derivatives markets. The payoff of the *lookback option* depends on the maximum of the underlying stock's price occurring over the life of the option. For instance, the European lookback call with fixed strike has the payoff:

$$\max(M_T - K, 0),$$

where the maturity is T and strike price is K.

1.1. **Stock Index Model.** We go through some essentials of stochastic financial modeling. If the reader is familiar with stochastic finance in continuous time, she or he could skip this subsection and move directly to subsection 1.2.

In the valuation of options and other derivatives contracts, it is important to properly express, by certain stochastic process, fluctuations of the underlying asset such as stock price, index, futures price, and interest rate. When it comes to modeling stock price movement, the best-known is arguably geometric Brownian motion, based on which the Black-Scholes formula is established. It says that the stock price $X = \{X_t; t \geq 0\}$ follows the dynamics:

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t, \quad X_0 = x, \tag{1.2}$$

where μ and σ are constant real numbers. The latter one is positive and called the *volatility* parameter. The initial value is x and the uncertainty is brought about by the Brownian motion W, which we explain below soon.

Note that (1.2) is called stochastic differential equation and has the explicit solution

$$X_t = X_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}. \tag{1.3}$$

Let us divide the both sides of (1.3) by X_0 and take the logarithm. Then $\log\left(\frac{X_t}{X_0}\right)$, which is the *log return* of the stock from time 0 to time t, follows *Brownian motion with drift* :

$$\log\left(\frac{X_t}{X_0}\right) = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t, \tag{1.4}$$

where the growth rate (=drift) is $g := \mu - \frac{1}{2}\sigma^2$. By discretizing the above equation into small time increments, we write

$$\log(X_{t+\Delta t}) - \log(X_t) = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma \cdot \varepsilon_t \sqrt{\Delta t} \tag{1.5}$$

where ε_t is a random variable following the standard normal distribution (mean 0 and variance 1) and does not depend on t. This is due to the property of Brownian motion. Accordingly, we write ε instead of ε_t. It reads that, from time t to time $t+\Delta t$, the log return $\log(X_{t+\Delta t}) - \log(X_t)$ in this interval Δt can be decomposed into $g\Delta t$ and $\sigma \cdot \varepsilon\sqrt{\Delta t}$. The former represents deterministic growth or decay and the latter is an approximation of the increment of Brownian motion. That is, $\mathrm{d}W_t \simeq \varepsilon\sqrt{\Delta t}$: during a small time interval Δt, Brownian motion increases (or decreases) approximately by $\varepsilon\sqrt{\Delta t}$. Hence we regard (1.5) as how we construct a sample path of the process (1.4): at time t we pick up a random number ε from the standard normal distribution and during this time interval Δt, the Brownian motion W is incremented by $\varepsilon\sqrt{\Delta t}$. Then we can compute, by adding the fixed value $g\Delta t$ and multiplying σ to $\varepsilon\sqrt{\Delta t}$, how much the log return changes. In the next time interval, we pick another number ε randomly and independently of the previous one and we compute the log return changes in this time interval. Then we move on to the next time step, and so on.

To begin with, we estimate the parameters μ and σ in (1.2) using daily data of the Nikkei 225 index from January 4, 2000 to January 31, 2022. For this purpose, we note that the right-hand side of (1.5) follows a normal distribution with mean $\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t$ and variance $\sigma^2\Delta t$: we have used the fact that ε follows the standard normal distribution. It follows that the mean and variance of $\log(X_{t+\Delta t}) - \log(X_t)$ are $\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t$ and $\sigma^2\Delta t$, respectively. It is thereby natural to compute the sample mean and sample variance of the time series

$$Y := \{\log(X_{t+\Delta t}) - \log(X_t), \quad \log(X_{t+2\Delta t}) - \log(X_{t+\Delta t}), \quad \log(X_{t+3\Delta t}) - \log(X_{t+2\Delta t}), \cdots\}.$$

Suppose that the sample mean and variance of Y are computed from data as a and b^2, respectively. Then we solve the equations:

$$a = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t, \quad b^2 = \sigma^2\Delta t$$

simultaneously for μ and σ. In our example, we obtain the estimates $\hat{\mu} = 0.043844$ and $\hat{\sigma} = 0.234460$, so that the growth rate is estimated as $\hat{g} := \hat{\mu} - \frac{1}{2}(\hat{\sigma})^2 = 0.016359$.

Equation (1.3) and (1.4) are included in a major class of stochastic processes, called a *linear* (or *one-dimensional*) *diffusion*. While we are not going into details of its mathematical properties, we wish to mention that linear diffusion is quite extensively employed in financial modeling.

1.2. **Renewing the record high?** Before we take up drawdowns, let us make a simple observation. The closing price of the Nikkei 225 on the January 31, 2022 was 27,001.98. Set this number as $x = 27,001.98$ and the record high price 38,915.87 as z. Let us define a random time

$$H_z := \min\{t \geq 0 : X_t = z\}, \tag{1.6}$$

which is the first time when the stock price X becomes level z. We are interested in the probability

$$F(t) := \int_0^t \mathbb{P}^x(H_z \in \mathrm{d}s).$$

The above equation for $F(\cdot)$ reads as follows: starting the Nikkei 225 at $x = 27,001.98$ and supposing that t is given, what is the probability that within t years the index recovers the historical record $z = 38,915.87$? For this computation, we can use the first passage time probability density of geometric Brownian motion (see Borodin and Salminen (2002)):

$$\mathbb{P}^x(H_z \in \mathrm{d}t) = \frac{|\log(z/x)|}{\sqrt{2\pi\sigma^2 t^3}} \left(\frac{z}{x}\right)^{\nu} \exp\left(-\frac{(\nu\sigma)^2 t}{2} - \frac{\log(z/x)^2}{2\sigma^2 t}\right) \mathrm{d}t. \tag{1.7}$$

where H_z is defined in (1.6) and $\nu := \frac{\mu}{\sigma^2} - \frac{1}{2}$.

Figure 2 plots (a) the probability density $\mathbb{P}^x(H_z \in \mathrm{d}t)$ and (b) the cumulative distribution function $F(t)$. Panel (a) shows that the distribution is the densest at $t = 0.8090$ years and getting sparse as time goes. Panel (b) tells us the following: within one year, the probability that the Nikkei 225 reaches the historical record is only 0.132487. That is, $F(1) = 0.132487$, if the parameters are $\mu = 0.043844$ and $\sigma = 0.234460$ for the next year: this probability depends on the starting value x, the parameter estimates $\hat{\mu}$ and $\hat{\sigma}$. We must keep in mind that the probabilities are computed based on the assumption that the parameters are constant during the time horizon of our analysis. Hence it is to be advised that one always use updated parameter estimates.

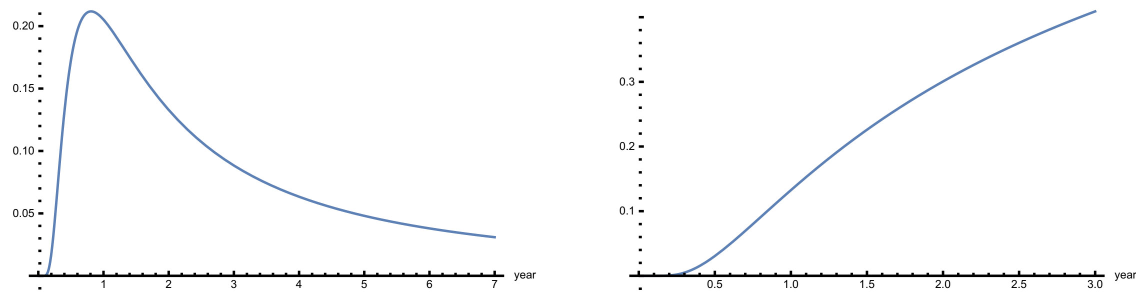
Recall that the growth rate estimate $\hat{g} = 0.016359$ is a positive number. This implies that the Nikkei 225 will *surely* be back to the all-time record of 38,915.87 if g is around the current level, while it may take long.

2. DRAWDOWN

Let us return to (1.1), that is,

$$Y_t := M_t - X_t,$$

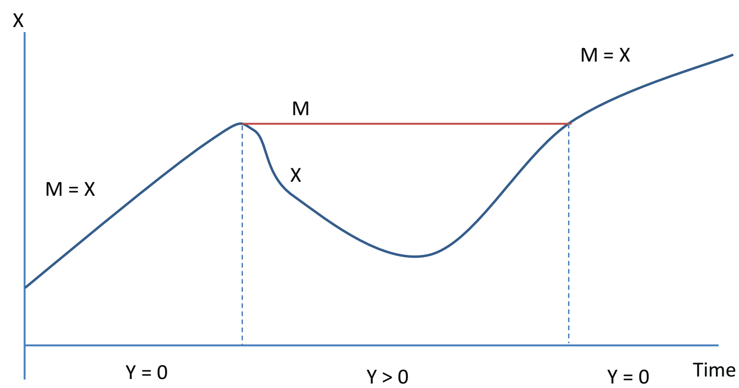
where $M_t = \max_{0 \leq s \leq t} X_s$. Recall that Y is the excursion, or drawdown of X from the running maximum M. The excursion process has a nice mathematical treatment which we shall utilize in this report as well. The first formal excursion theory appeared in Itô (1970). The book by Zhang (2018) includes a lot of useful computations involving drawdowns. Figure 3 is a schematic diagram about what has been



(a) (b)

Fig. 2. The first passage time to the record high: (a) The probability density function $\mathbb{P}^x(H_z \in \mathrm{d}t)$ and (b) The cumulative distribution function $\int_0^t \mathbb{P}^x(H_z \in \mathrm{d}s)$.

Fig. 3. Schematic presentation of a drawdown: The blue line is X, which is identical to M when the running maximum is renewed: $Y = M - X = 0$ in this period. The red flat line of M is in the period $Y = M - X > 0$, so that our stock is in a drawdown period.



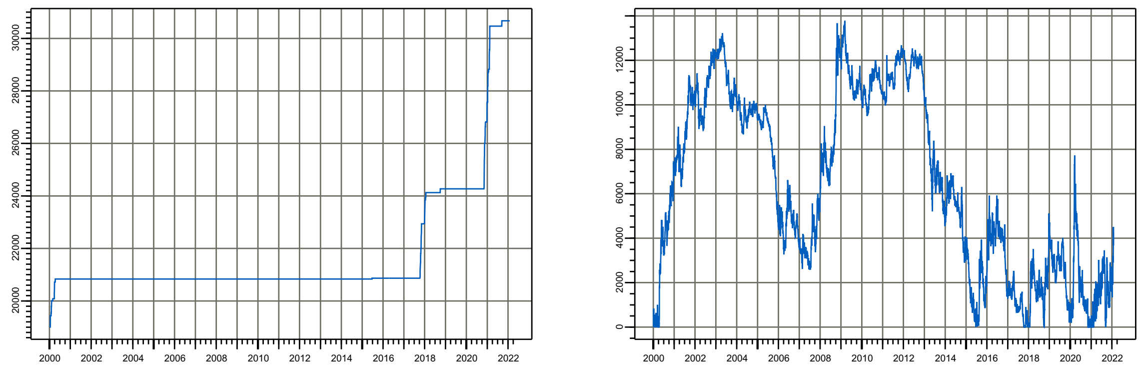
explained. The blue line is X. This is identical to M when X keeps increasing and therefore the running maximum is renewed: $Y = M - X = 0$ in this period. There is no drawdown. On the other hand, the red flat line of M indicates the period of $Y = M - X > 0$, so that our stock is in a drawdown period. It is in the middle part of the diagram. We present in Figure 4 (a) the running maximum M and (b) excursions from the running maximum $Y = M - X$ of the daily Nikkei 225 index from January 4, 2000 to January 31, 2022.

If the corporate value is supposed to increase over time, we should measure performance of the company's stock with reference to its running maximum. When we take this view, the excursion theory is helpful. In fact, one can calculate interesting and useful quantities concerning the running maximum of stock price or stock return.

Finally, linear diffusion process has its own *scale function*, denoted by $s(\cdot)$. In the case of geometric Brownian motion in (1.2), it is

$$s(x) = \begin{cases} -\frac{x^{-2\nu}}{2\nu}, & \nu \neq 0, \\ \log x, & \nu = 0. \end{cases} \tag{2.1}$$

where $\nu = \frac{\mu}{\sigma^2} - \frac{1}{2}$ as above.



(a) (b)

Fig. 4. The Nikkei 225 daily data from January 2000 to January 2022: (a) The running maximum M and (b) excursions from the running maximum $M - X$.

2.1. **First problem.** Now that we have introduced the scale function, let us perform a quick computation. The defining property of the scale function is

$$\mathbb{P}^x(T_m < T_\ell) = \frac{s(x) - s(\ell)}{s(m) - s(\ell)}, \quad \ell < x < m, \tag{2.2}$$

where $s(\cdot)$ is defined in (2.1). Imagine that the current position x of the process is somewhere in an interval (l, m). Then, (2.2) is the probability that the process comes out of the interval from the right-end, m. In the context of stock investment, if the investor sets the mind that she/he would sell the stock when the price attains either the target m or down to the stop loss level l, whichever occurs earlier, the simple equation (2.2) provides the probability of the investor finishing this investment successfully. A quick example is as follows: the current price x is $27,001.98$ as of January 31, 2022, the target level is $m = 29,702.18$, and stop-loss level is $\ell = 25,651.88$, so that the investor wishes to earn 10% return before losing 5%. Using the estimated parameters of drift $\hat{\mu}$ and volatility $\hat{\sigma}$ of the Nikkei 225, the odds are below $1/2$, which is 0.35984.

References

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