A FINANCIAL ENGINEERING VIEW OF DRAWDOWNS IN THE STOCK MARKET PART II

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ABSTRACT. In this report, we collect some topics about how the stochastic (probabilistic) model analysis can provide us with insight into stock investment. We focus on drawdowns from the running maximum. Suppose that the stock price starts at time 0, and after a certain time interval t, the highest price between 0 and t is attained at s. The price at s is called the running maximum up to time t. It follows that the current price, which is the price at time t, is lower than the running maximum. The stock is said to be in a *drawdown* period (from the running maximum).

Following Part I where we discussed financial modeling for the drawdown process, in this Part II, we keep the Nikkei 225 index as our example and look into two more problems of interest, emphasizing drawdown and recovery to the running maximum. We attempt to avoid technical details in favor of intuitive understanding.

2. DRAWDOWN (CONTINUED)

Recall the drawdown at time t is defined

$$Y_t := M_t - X_t,$$

where $M_t = \max_{0 \le s \le t} X_s$. We consider two problems concerning the size and length of drawdowns to see how these notions are related to stock investment.

2.2. Problem concerning with the size of drawdown.

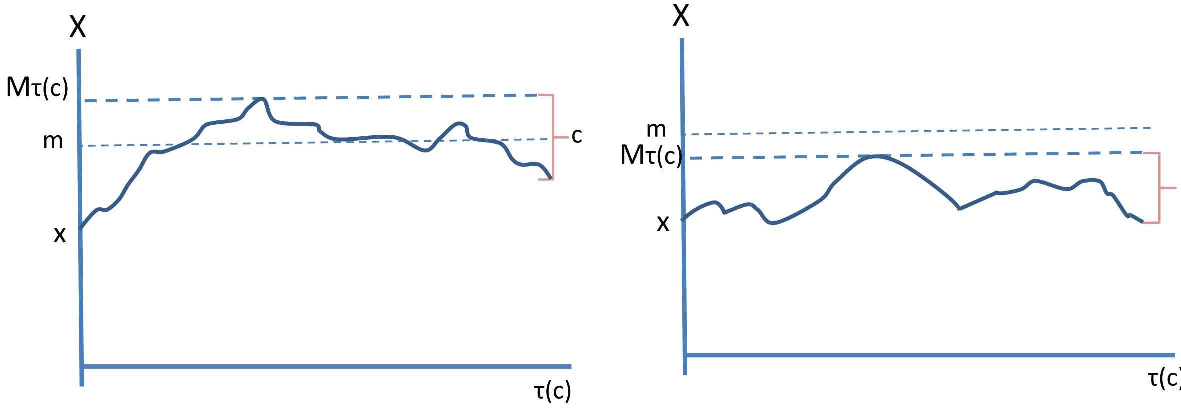
Let us define another random time:

$$\tau(c) := \inf\{t > 0 : M_t - X_t > c\},$$

which is the first time when the drawdown exceeds c. This should not be confused with the first passage time of X, which was introduced in Part I. Let us compute a little bit complicated probability: given $m > x$ and $c > 0$,

$$\mathbb{P}^x(M_{\tau(c)} > m), \tag{2.3}$$

which reads in the following way. Suppose that today's ($t = 0$) stock price is x at which the investor buys and the investor sets the target at level m. Then, (2.3) is the probability that, before the drawdown of size c occurs, the highest stock price recorded from time 0 up to that time $\tau(c)$, denoted by $M_{\tau(c)}$, is greater than m. In other words, the investor tolerates the drawdown (from the then highest price) up to size c. But until the time of a first drawdown of size c, she or he wishes to have attained the level m. This

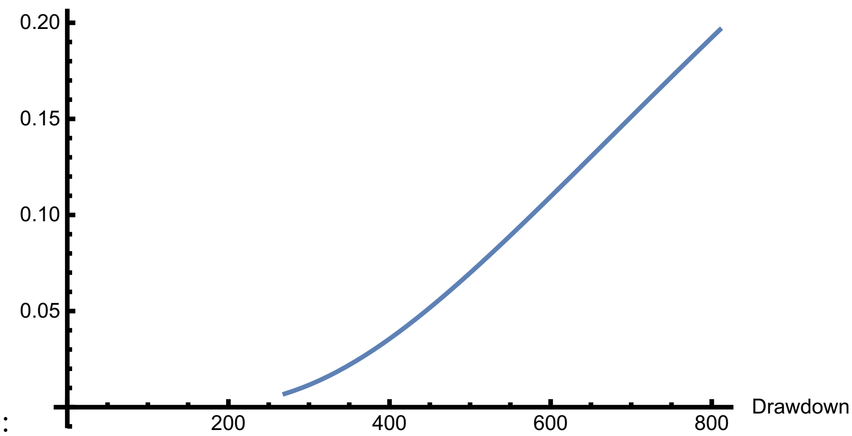


(a)

(b)

Fig. 1. Schematic presentation of $M_{\tau(c)} > m$: (a) Successful path: the running maximum was attained at $M_{\tau(c)}$ and this value is greater than m. (b) Unsuccessful path: the running maximum was attained at $M_{\tau(c)}$ and this value is less than m.

Fig. 2. The probability of Equation (2.4) with $x = 27,001.98$, the closing price on January 31, 2022 and the target $m = 28,352.1$ which accounts for 5% return, while the tolerable drawdown size c varies from 270 to 810, corresponding to from 1% to 3% of the current price index.



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strategy allows the investor to pursue higher prices until time $\tau(c)$, and at the same time to avoid regrets in a situation where the price would become higher after she or he sells at a mediocre price. Note that the investor can set the size of tolerable drawdown c at her or his own choice. Now the selling price is $M_{\tau(c)} - c$. By setting m high enough, the investor may end up with earning capital gain if $M_{\tau(c)} - c > x$. See Figure 1 for a schematic diagram of this problem. We illustrate both successful and unsuccessful realizations in Panels (a) and (b), respectively. $M_{\tau(c)}$ is greater than m in Panel (a) but vice versa in Panel (b).

In fact, an explicit form of (2.3) is available in Lehoczky (1977):

$$\mathbb{P}^x(M_{\tau(c)} > m) = \exp\left(-\int_x^m \frac{s'(u)}{s(u)-s(u-c)}\mathrm{d}u\right) \tag{2.4}$$

where the scale function $s(\cdot)$ of geometric Brownian motion is given in Part I and reproduced here:

$$s(x) = \begin{cases} -\frac{x^{-2\nu}}{2\nu}, & \nu \neq 0, \\ \log x, & \nu = 0, \end{cases} \tag{2.5}$$

where $\nu := \frac{\mu}{\sigma^2} - \frac{1}{2}$. See Egami and Oryu (2017) for the discounted version of (2.4), namely $\mathbb{E}^x[e^{-qH_m} : M_{\tau(c)} > m]$ which takes into account the time value until the maximum level m is attained with q being an appropriate discount rate.

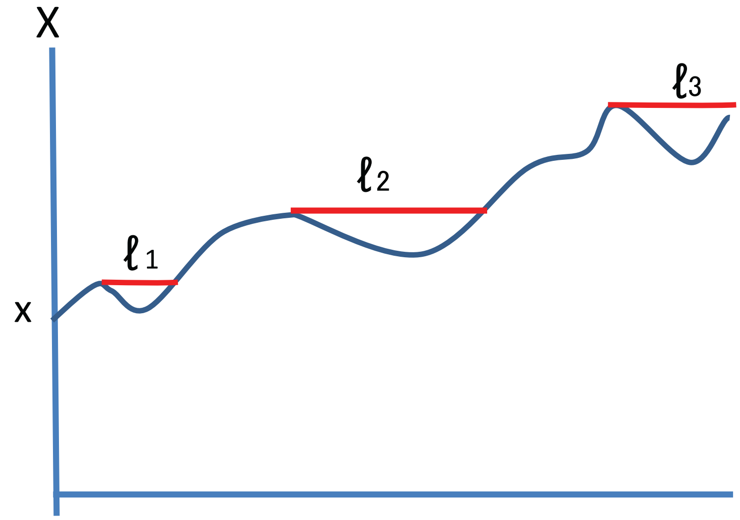
We plot, in Figure 2, the probability (2.4) by fixing the purchase price at $x = 27,001.98$, which is the closing Nikkei 225 index on January 31, 2022 and the target $m = 28,352.1$ which accounts for 5% return, while the tolerable drawdown size c varies from 270 to 810, corresponding to from 1% to 3% of the purchase price. We use the parameter estimates of $\hat{\mu} = 0.043844$ and $\hat{\sigma} = 0.234460$ computed in Part I. Note that the estimate of growth rate is $\hat{g} = \hat{\mu} - \frac{1}{2}(\hat{\sigma})^2 = 0.016359$. It is natural to observe that if one allows for a larger drawdown, then the probability of winning $m = 28,352.1$ before τ_c becomes greater. It is because one can wait longer. According to the result, the probability of winning is small in this case: 0.19639 even if the investor prepares himself for $c = 810$.

2.3. **Problem concerning the length of drawdown.** Next, we consider the *duration* or length of drawdowns. It is also known as *time to recover* the historical running maximum. As explained at the beginning of this section, there appears an excursion, or drawdown from the running maximum if $Y_t = M_t - X_t > 0$ at time $t \geq 0$. When X comes back to the running maximum at time $u > t$, we have $Y_u = 0$ since $M_u = X_u$. Then the duration is defined as the time interval from the beginning and to the end of an excursion. We shall use the terms of duration and length, interchangeably. A schematic diagram of this notion is Figure 3. The blue curve is a sample path of stock price X. When X keeps renewing the running maximum, we have $M = X$. When $M > X$, the running maximum remains the same during the drawdown. The flat red lines indicate the unrenewed maximum values M. The durations of these drawdowns are indicated by ℓ_1, ℓ_2, and ℓ_3. We shall use ℓ for this random variable. See Getoor (1979) and Rogers and Williams (1994) [chapter VI] for theoretical details.

Let us consider what we could tell about the distribution of the length ℓ: this information might help the investor set up, with proper timing, her or his buy-and-sell cycle. Let us define a function $h(u)$ for $u \in (0, \infty]$. Notice that u could take the value of infinity. This h-function indicates the rate at which a *first* drawdown of length ℓ greater than u occurs. To be more concrete, define the random variable Γ^u as the first time that a drawdown of length greater than u appears. Then for $t > 0$, we have

$$\mathbb{P}(\Gamma^u > t) = e^{-h(u)t}. \tag{2.6}$$

Fig. 3. Schematic presentation of the duration of drawdowns :The blue curve is a sample path of stock price X. As X keeps renewing the running maximum, we have $M = X$. When $M > X$, we denote, by the red lines, the running maximum unchanged during the drawdown. The lengths (durations) of these drawdowns are indicated by ℓ_1, ℓ_2, and ℓ_3.



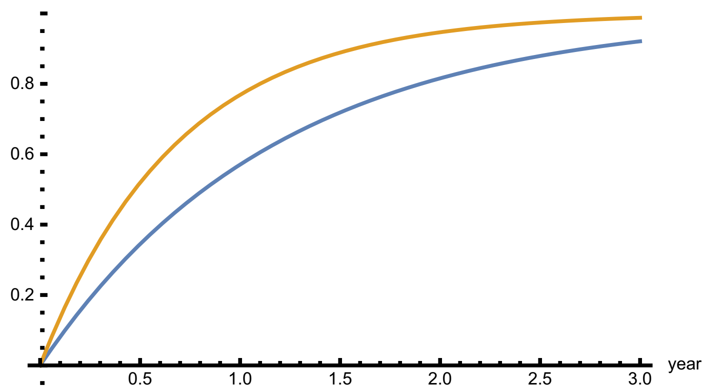
Note that there may appear drawdowns before time Γ^u. By the definition of Γ^u, the length of these drawdowns should be less than u. In measuring Γ^u, the lengths of these drawdowns are *not* included.[1] This is a consequence of the underlying mathematics, but it is practically simpler than otherwise. As said, u is allowed to take infinity. A drawdown of infinite length means the case where the stock price X shall not return to the most recent running maximum, forever.

Let us consider the log return this time for variety instead of price itself while the probabilities to be computed below are the same. It follows that we look at Brownian motion with drift described in (1.4) in Part I. See section 3 in which we discuss Brownian motion with drift.

Fix $u = \frac{1}{12}$ and $\frac{1}{4}$: we are interested in a cycle of length greater than a month and a quarter of the year. An explicit formula for $h(\cdot)$ is available in this case (see subsection 3.2). Recall that the estimate of growth rate $\hat{g}$ is $0.016359 > 0$. As you may imagine, however, when the growth rate g is positive even if it is a small number, the process tends to go upwards and keep renewing running maximums instantly. Hence it is not practical to consider this case. Accordingly, for the purpose of illustration, by slightly changing μ to a hypothetical value of 0.02 and keeping $\sigma = 0.234460$ intact, we would have a negative growth rate of $g = -0.007486$. As Figure 4 shows the cumulative distribution of $\Gamma^{\frac{1}{12}}$ (orange line) and $\Gamma^{\frac{1}{4}}$ (blue line). The graphs read, for example, that the probabilities that a first drawdown of length greater than $1/12$ or $1/4$ appears within one year is 0.7688 and 0.5706, respectively. The average waiting time

[1] For example, suppose that we are interested in a drawdown of length 1 and there are two drawdowns of length 0.3 and 0.5 before the first drawdown of length greater 1 appears. In this case, the t in (2.6) does not include 0.3 and 0.5. Remember that these numbers 0.3 and 0.5 are random: we cannot foresee the length of these at time 0.

Fig. 4. The cumulative distribution of $\Gamma^{\frac{1}{12}}$ (orange line) and $\Gamma^{\frac{1}{4}}$ (blue line). For example, the probabilities that a first drawdown of length greater than $1/12$ or $1/4$ appears within one year is 0.7688 and 0.5706, respectively.



for a first drawdown of length greater than $1/12$ and $1/4$ can be computed as 0.6829 years and 1.1828 years, respectively. It seems to take long time before observing these drawdowns, relative to $1/12$ and $1/4$. Be advised, again, that these years exclude the durations of drawdowns that might appear before $\Gamma^{\frac{1}{12}}$ or $\Gamma^{\frac{1}{4}}$.

Since $g = -0.007486$ is close to zero, log return, $\log(X_t/x)$, behaves like a Brownian motion with no drift. So, let us think about Brownian motion $W = (W_t, t \geq 0)$ starting at the origin. The mean value of W_t is 0 for any $t \geq 0$, so that, heuristically, there is a great chance that W returns to the origin many times. In terms of excursion from the running maximum, it corresponds to the situation where $\log(X_t/x)$ starts at the current maximum, say $M_0 = s$, and is expected to return to s in a short period: the duration is therefore short on average. This is a possible explanation why a drawdown of length greater than $1/12$ or $1/4$ does not seem to appear easily.

3. APPENDIX

We collect the functionals that are concerned with Brownian motion with drift. Let us define $Y = (Y_t, t \geq 0)$ as follows:

$$Y_t := y + gt + \sigma W_t, \quad Y_0 = y, \tag{3.1}$$

where g is a real number, $\sigma > 0$, and $W = (W_t, t \geq 0)$ is a standard Brownian motion. The formulae given in this section are for this Y. The scale function for Y is as follows:

$$s(y) = \frac{\sigma^2}{2g}\left(1 - \exp\left(-\frac{2g}{\sigma^2}y\right)\right).$$

Note that setting $\sigma = 1$ and letting $g \to 0$, we obtain $s(y) = y$ for all $y \in (-\infty, \infty)$, which is the scale function of a standard Brownian motion W.

3.1. **The first passage time density.** Then the first passage time density to $z \in \mathbb{R}$ is

$$\mathbb{P}^y(H_z \in \mathrm{d}t) = \frac{|z-y|}{\sqrt{2\pi\sigma^2 t^3}} \exp\left(-\frac{(z-y-gt)^2}{2\sigma^2 t}\right) \mathrm{d}t. \tag{3.2}$$

where H_z is the first passage time of Y to point z from point y.

3.2. **Length of Drawdown.** Following Proposition 7.22 of Getoor (1979), we compute the $h(\cdot)$ function for Brownian motion with drift. In fact, $h(\cdot)$ is implicitly represented as

$$\frac{1}{\alpha}\left(\frac{\sigma^2}{2\sqrt{2\alpha^2\sigma^2+g^2}} \exp\left(-\frac{2g}{\sigma^2}x\right)\right)^{-1} = \int_0^\infty e^{-\alpha s} h(s) \mathrm{d}s$$

for $x \in (-\infty, \infty)$. We can invert this Laplace transform by Mathematica™ for an explicit form of $h(\cdot)$:

$$h(s) = 2\exp\left(\frac{2g}{\sigma^2}x\right)\left[\frac{\sqrt{\frac{2}{\pi}}\exp(-\frac{g^2 s}{2\sigma^2})}{\sigma\sqrt{s}} + \frac{g\cdot \mathrm{erf}\left(\frac{g\sqrt{s}}{\sqrt{2\sigma^2}}\right)}{\sigma^2}\right]$$

where $\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\mathrm{d}t$ is the Gauss error function.

References

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