THE STRONG MARKOV PROPERTY APPLIED TO OPTION PRICING PART II

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ABSTRACT. In pricing and hedging derivatives, one uses stochastic processes to represent movement of underlying assets and other important state variables (such as volatility). In this report, we collect several techniques from stochastic analysis which we believe are useful for financial modeling, derivative pricing, and parameter estimation. A key element is the strong Markov property that helps to streamline seemingly complex equations and to tremendously reduce variance in numerical simulations.

Following Part I where we discussed the definition of the strong Markov property and related mathematical objects, we show more examples and a case where the strong Markov property is not applicable. We include proofs of some technical, but useful for applications, results in the Appendix.

4. OTHER EXAMPLES

Let us keep our general setting in Part I and consider a set of all the possibilities of future outcomes Ω and denote our stochastic process by $X = \{X_t; t \geq 0\}$ with the initial value x. In this report, we assume that the process X is *right continuous*. Given a future scenario $\omega \in \Omega$ and a fixed time t, the stochastic process X takes a value $X_t(\omega)$ in some space (such as $\mathbb{R}^d$). Given this future path ω, X keeps taking values as time goes and a history of its behavior is recorded. This is called the *filtration* generated by X and denoted by $\mathscr{F} = \{\mathscr{F}_t; t \geq 0\}$. Intuitively, $\mathscr{F}_t$ is the accumulated information of X up to time t. Recall that a random time U is called a stopping time of $\mathscr{F}$ if we can tell whether or not U has happened up to time t by looking at the record of X up to time t: $\{\omega : U(\omega) \leq t\} \in \mathscr{F}_t$ for each t.

If $\{X_t, \mathscr{F}_t, t \geq 0\}$ is a *strong Markov* process, it has the property for a certain set A in $\mathbb{R}^d$

$$\mathbb{P}_x(X_{U+t} \in A | \mathscr{F}_U) = \mathbb{P}_x(X_{U+t} \in A | X_U),$$

where $\mathscr{F}_U$ is the record of X up to the stopping time U.

For concrete computations and for practical applications, let $X = (X_t, t \geq 0)$ be a diffusion process taking values on $\mathbb{R}^d$. The shift operator $\theta_s : \Omega \to \Omega, s \geq 0$ further facilitates rigorous argument:

$$X_{s+t}(\omega) = X_t(\theta_s(\omega)) = (X_t \circ \theta_s)(\omega) \quad \text{for each } s, t \geq 0.$$

The infinitesimal generator $\mathfrak{G}$ of a diffusion X is defined by

$$\mathfrak{G}g(x) := \lim_{t \downarrow 0} \frac{\mathbb{E}_x[g(X_t)] - g(x)}{t} \tag{4.1}$$

for a suitable function g. This is how much the value $g(X_t)$ would change if an infinitely small amount of time has elapsed from time zero. The infinitesimal generator for a linear diffusion $\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$ is

$$\mathfrak{G}g(x) = \frac{1}{2}\sigma^2(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}g(x) + \mu(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x), \tag{4.2}$$

Now we shall deal with two random times. In Part I, we denote the first time that X reaches b by $H_b := \min\{t \geq 0 : X_t \leq b\}$, which is a good example of stopping time. A generalization of this is, for a Borel set A,

$$T_A := \inf\{t > 0 : X_t \in A\}$$

where $\inf\emptyset = +\infty$ by convention. This is called the *first hitting time*. If the Markov process is right continuous, it is a stopping time (see Theorem 2.4.5 of Chung and Walsh (2004)).

If $\sigma \leq T_A$ where σ is a stopping time and T_A is a first hitting time to a Borel set A, then according to Chapter III.7 of Peskir and Shiryaev (2006)

$$T_A = \sigma + T_A \circ \theta_\sigma. \tag{4.3}$$

For example, with (4.3) the strong Markov property provides

$$\begin{aligned}
\mathbb{E}_x[e^{-\alpha T_A}\mathbb{1}_{\{\sigma\leq T_A\}}] &= \mathbb{E}_x[e^{-\alpha(\sigma+T_A\circ\theta_\sigma)}\mathbb{1}_{\{\sigma\leq T_A\}}] \\
&= \mathbb{E}_x[\mathbb{E}_x[e^{-\alpha(\sigma+T_A\circ\theta_\sigma)}\mathbb{1}_{\{\sigma\leq T_A\}}|\mathscr{F}_\sigma]] \\
&= \mathbb{E}_x[e^{-\alpha\sigma}\mathbb{E}_x[e^{-\alpha(T_A\circ\theta_\sigma)}\mathbb{1}_{\{\sigma\leq T_A\}}|\mathscr{F}_\sigma]] \\
&= \mathbb{E}_x[e^{-\alpha\sigma}\mathbb{E}_{X_\sigma}[e^{-\alpha T_A}\mathbb{1}_{\{\sigma\leq T_A\}}]],
\end{aligned}$$

so that we may use the information about X at time σ.

The relation (4.3) may remind one of the *memoryless property* of an exponential random variable. Let ν be an exponential random variable with rate λ (i.e., $\mathbb{E}[\nu] = 1/\lambda$). Then we can write

$$\nu = t + \nu \circ \theta_t \quad \text{if } \nu > t. \tag{4.4}$$

Suppose that the next bus arrives after an exponentially distributed random time of rate λ and there is a person who has already waited for a t amount of time. But at time t, this person's average waiting time for the bus is still $1/\lambda$: the fact that the passenger has waited for the t amount of time is forgotten (in future scenarios). If we write (4.4) as $\nu - t = \nu \circ \theta_t$, additional waiting time over t (the left-hand side) is equal to ν reckoned from time t (the right-hand side).

The equations (4.3) and (4.4) are useful, for example, in studying options with multiple exercising rights (see Carmona and Dayanik (2008)) and financial instruments under regime switching models (see Hamilton (2008)). Regime switching is widely used since it allows model parameters to alter according to economic conditions. A switch from one regime to another is often assumed to be governed by a Poisson process, so that the waiting time before switching follows exponential distribution. In case there are two regimes (e.g. bulls and bears), X takes the values of 1 or 2, so that we consider two-state Markov chain with the infinitesimal generator

$$\begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix} \tag{4.5}$$

where λ_1 is the rate of waiting time of a jump from state 1 to state 2 and λ_2 is the rate of waiting time of a jump from state 2 to state 1. While the appearance of this generator is very different from (4.2) (linear diffusion case), it can be shown this is so. See Appendix B for a derivation of this matrix from the definition (4.1).

5. Beyond the strong Markov property

The key element of the strong Markov property and its applications is that the random time at which the process is evaluated has to be a stopping time. Recall that a random time U is called a stopping time of $\mathscr{F}$ if we can tell whether or not U has happened up to time t by looking at the record of X up to time t: that is, $\{\omega : U(\omega) \leq t\} \in \mathscr{F}_t$ for each t. The first hitting time is a good example of stopping times. On the other hand, the *last exit time* is a typical example of a random time that is *not* a stopping time. Suppose that a Markov process $X = \{X_t : t \geq 0\}$ is *transient*. This means heuristically that the lifetime, say ζ, of X is finite, so that X disappears at some point of time.[1]

For a Borel set B, the last exit time from B is defined by

$$L_B(\omega) := \sup\{t \geq 0 : X_t(\omega) \in B\}$$

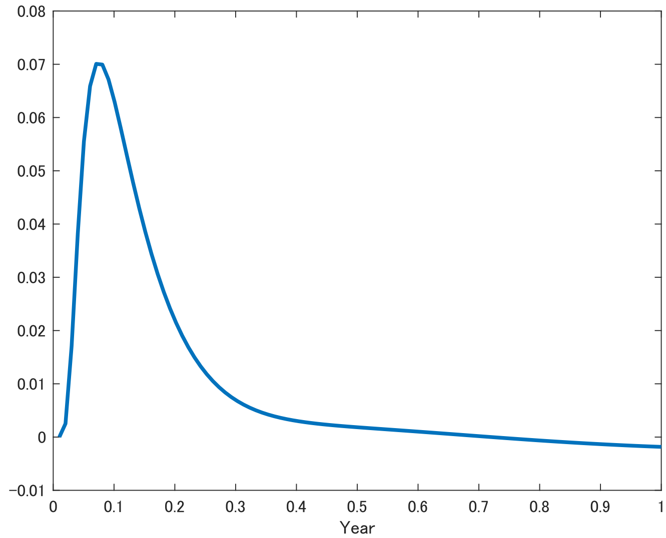
where $\sup\emptyset = 0$ by convention. Since X is assumed to be transient, it disappears at some time and the set $\{t \geq 0 : X_t \in B\}$ is finite. Suppose that X exits from B at time s. But we cannot tell at that time s whether this is the last time for the process to exit from B. Hence L_B is not a stopping time and we cannot discuss the strong Markov property at L_B. Despite of this difficulty, the last exit time is increasingly applied in the finance literature as indicated in Nikeghbali and Platen (2013).

Consider an investment in the Nikkei 225. Suppose that the investor sets a stop-loss level at c and an alarming price level at $\alpha(> c)$ since he needs to prepare well in advance for a possible loss. The investor models the Nikkei 225 by a geometric Brownian motion X and wishes to estimate how long it would take from the *last* exit time from level α to reaching the stop-loss threshold c. He needs this information perhaps for some liquidity concerns when loss seems inevitable. In this case the set B in the above definition is a singleton set $\{\alpha\}$. In sum, the investor wishes to compute

$$\mathbb{E}_x[H_c - L_\alpha].$$

[1] In financial engineering this concept of transience is used, for example, in barrier options of *up-and-out* and *down-and-out* types. A down-and-out option becomes worthless once the underlying asset hits a threshold level $b(< x)$. X is considered to disappear at the first time when X hits b.

Fig. 3. THE DENSITY OF THE DISTRIBUTION $\mathbb{P}[H_c - L_\alpha \in \mathrm{d}t : H_c < \infty]$.



Since H_c and L_α are closely related random times, it is not easy to efficiently compute the distribution even by simulations. To overcome this difficulty, Egami and Kevkhishvili (2020) employs the *reversed process*: the original process X reversed from the time H_c. Let it be called Z which starts at c. The infinitesimal generator of the reversed process Z is identified (see Corollary 3.8 of the paper). Then the quantity $(H_c - L_\alpha)$ with respect to X is the same as the first hitting time to α of Z. Hence the problem transforms into computing the distribution of the first hitting time of Z. Note that the result of the said paper is expressed in terms of a Brownian motion with non-zero drift and unit variance, so that we need to transform a geometric Brownian motion $X_t = xe^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$ to

$$\frac{1}{\sigma}\log\left(\frac{X_t}{x}\right) = \left(\frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\right)t + W_t,$$

where $W = \{W_t : t \geq 0\}$ is a Brownian motion.

Let us take a hypothetical situation where $c = 30{,}000$ and $\alpha = 35{,}000$ for the Nikkei 225. The parameters are set to be $\sigma = 0.235$ and $\mu - \frac{1}{2}\sigma^2 = -0.125$. Note that an estimation procedure of the parameters is discussed in Egami (2022). In this case, X reaches the level c in finite time with probability one. Figure 3 shows the probability density $\mathbb{P}[H_c - L_\alpha \in \mathrm{d}t : H_c < \infty]$. The parameters are standard when the market is slightly bearish. The density function has the sharp peak around $t = 0.08$, indicating that the time interval between the last exit from level α and the stop-loss level c is rather short. This kind of information could be useful from the risk management point of view. Moreover, when one needs to evaluate options of a look-back type in relation to a last exit time, the concept of reversed process can be helpful.

Appendix A. Derivation of (3.3) in Part I

Only the second term of (3.3) needs a proof. The derivation of

$$\mathbb{E}_x\left[\int_T^\infty e^{-\alpha t} f(X_t)\mathrm{d}t\right] = \mathbb{E}_x[e^{-\alpha T}U^\alpha f(X_T)].$$

is done by making substitution $t = T + u$ as below.

$$\begin{aligned}\mathbb{E}_x\left[\int_T^\infty e^{-\alpha t} f(X_t)\mathrm{d}t\right] &= \mathbb{E}_x\left[\mathbb{E}_x\left\{\int_T^\infty e^{-\alpha t} f(X_t)\mathrm{d}t\middle|\mathscr{F}_T\right\}\right]\\ &= \mathbb{E}_x\left[\mathbb{E}_x\left\{\int_0^\infty e^{-\alpha (T+u)} f(X_{T+u})\mathrm{d}u\middle|\mathscr{F}_T\right\}\right]\\ &= \mathbb{E}_x\left[e^{-\alpha T}\mathbb{E}_x\left\{\int_0^\infty e^{-\alpha u} f(X_{T+u})\mathrm{d}u\middle|\mathscr{F}_T\right\}\right].\end{aligned}$$

Let us consider $\int_0^\infty e^{-\alpha u} f(X_u)\mathrm{d}u$ as a random variable, say Y. With this in mind, we can continue

$$\begin{aligned}\mathbb{E}_x\left[e^{-\alpha T}\mathbb{E}_x\left\{\int_0^\infty e^{-\alpha u} f(X_{T+u})\mathrm{d}u\middle|\mathscr{F}_T\right\}\right] &= \mathbb{E}_x\left[e^{-\alpha T}\mathbb{E}_x\left\{\int_0^\infty e^{-\alpha u} f(X_u\circ\theta_T)\mathrm{d}u\middle|\mathscr{F}_T\right\}\right]\\ &= \mathbb{E}_x\left[e^{-\alpha T}\mathbb{E}_x\left\{\left(\int_0^\infty e^{-\alpha u} f(X_u)\mathrm{d}u\right)\circ\theta_T\middle|\mathscr{F}_T\right\}\right]\\ &= \mathbb{E}_x\left[e^{-\alpha T}\mathbb{E}_x\left\{Y\circ\theta_T|\mathscr{F}_T\right\}\right]\\ &= \mathbb{E}_x\left[e^{-\alpha T}\mathbb{E}_{X_T}\left\{Y\right\}\right]\\ &= \mathbb{E}_x\left[e^{-\alpha T}\mathbb{E}_{X_T}\left\{\int_0^\infty e^{-\alpha u} f(X_u)\mathrm{d}u\right\}\right] = \mathbb{E}_x\left[e^{-\alpha T}U^\alpha f(X_T)\right]\end{aligned}$$

by using (3.2) in Part I for the fourth line and the definition of $U^\alpha f(\cdot)$ in the fifth line.

Appendix B. The generator of a Markov chain

Let us recall an indicator function for a set A

$$\mathbb{1}_A(x) = \begin{cases} 1, & \text{if } x\in A,\\ 0, & \text{otherwise.}\end{cases}$$

Our state variable X takes the value of 1 or 2. First we show why the (1,2) entry of the matrix (4.5) is λ_1. The (1,2) entry corresponds to the situation where X moves from state 1 to state 2 within a small amount of time. Now X is initially in state 1 and we write the generator according to the definition (4.1):

$$\begin{aligned}\lim_{t\downarrow 0}\frac{\mathbb{E}[\mathbb{1}_{\{\text{not in state 1}\}}(X_t)] - \mathbb{1}_{\{\text{not in state 1}\}}(x)}{t} &= \lim_{t\downarrow 0}\frac{\mathbb{P}(X_t\neq 1) - 0}{t}\\ &= \lim_{t\downarrow 0}\frac{\mathbb{P}(T_1 < t)}{t} = \lim_{t\downarrow 0}\frac{1-e^{-\lambda_1 t}}{t} \simeq \frac{\lambda_1 t}{t} = \lambda_1\end{aligned}$$

where we apply $g(x) = \mathbb{1}_A(x)$ in (4.1) and $A = \{\text{not in state 1}\}$. T_1 denotes the waiting time until a jump to state 2 from state 1 and is exponentially distributed with rate λ_1.

Next the (1,1) entry corresponds to the situation where X starts in state 1 and remains the same within a small amount of time. Similar to the case of the (1,2) entry,

$$\lim_{t\downarrow 0}\frac{\mathbb{E}[\mathbb{1}_{\{\text{stay in state 1}\}}(X_t)]-\mathbb{1}_{\{\text{stay in state 1}\}}(x)}{t}=\lim_{t\downarrow 0}\frac{\mathbb{P}(X_t=1)-1}{t}$$
$$=\lim_{t\downarrow 0}\frac{\mathbb{P}(T_1>t)-1}{t}=\lim_{t\downarrow 0}\frac{e^{-\lambda_1 t}-1}{t}\simeq\frac{-\lambda_1 t}{t}=-\lambda_1,$$

which proves that the (1,1) entry of (4.5) is $-\lambda_1$.

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